Inductive theorem proving based on tree grammars

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Introduction

- Invariants are (often) non-analytic
  How to find key lemma?
- Consider proofs of instances $A(n)$ of $\forall x\, A(x)$
  E.g. bounded model checking, constructive ω-rule, ...
- Proof-theoretic analysis of structure of instance-proofs
  - Based on Herbrand's theorem and tree grammars
  - Extending method for cut-introduction to induction

Outline

$\Rightarrow$ Cut-elimination and cut-introduction

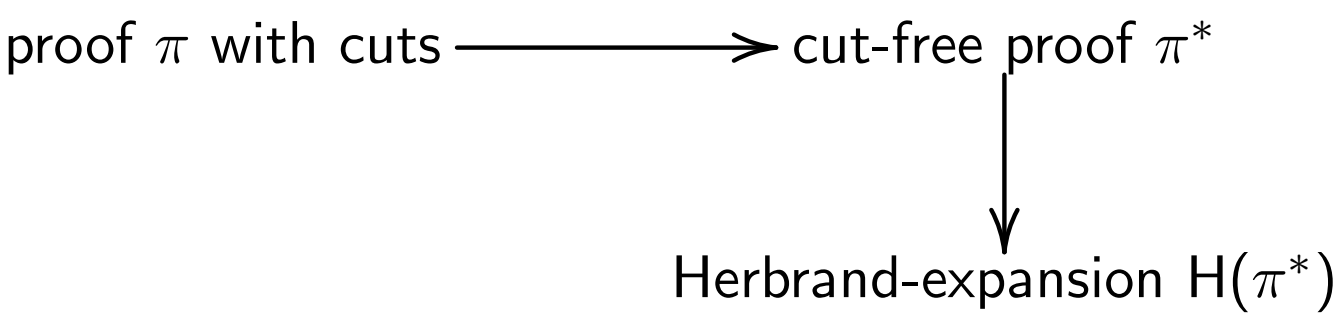
- Inductive theorem proving based on tree grammars
- Example

Cut-elimination and Herbrand's theorem

- The cut rule:

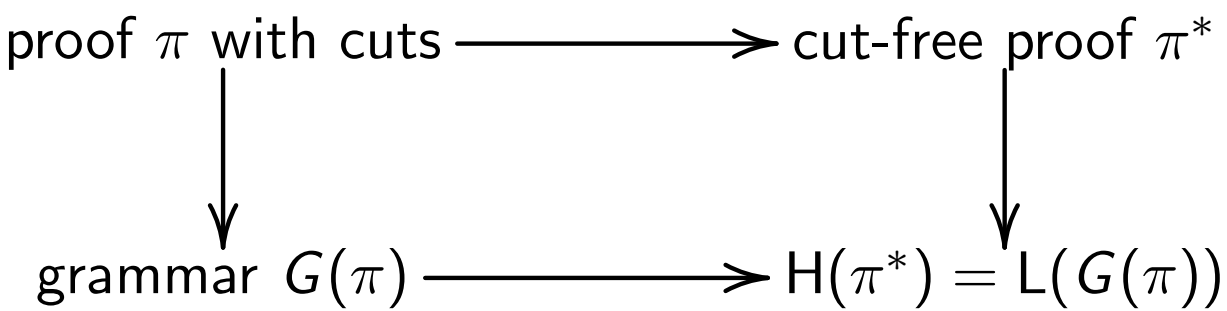
$$\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ \text{cut}$$

- **Cut-elimination theorem** (Gentzen 1934). Every proof can be transformed into a proof without cuts.
- **Herbrand's theorem** (1930). A first-order formula φ is valid iff a finite expansion of φ is a tautology.
  E.g. $\exists x\, A_{\text{qf}}(x)$ expands to $A_{\text{qf}}(t_1) \vee \ldots \vee A_{\text{qf}}(t_n)$
- In total:

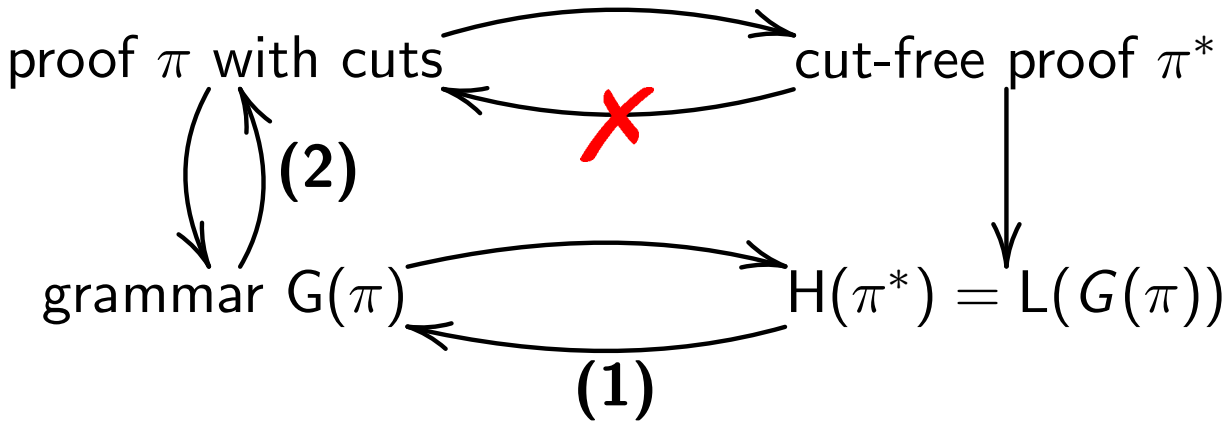


Cut-elimination and tree grammars

- Information of Herbrand-expansion $A_{\text{qf}}(t_1) \vee \ldots \vee A_{\text{qf}}(t_n)$ is $T = \{t_1, \ldots, t_n\}$ – a (finite) tree language.
- **Theorem** (H 2012). If π is a proof with Π_1-cuts then there is a totally rigid acyclic tree grammar $G(\pi)$ s.t. $L(G(\pi))$ is a Herbrand-expansion and $|G(\pi)| \leq |\pi|$.
- In total:



Cut-introduction based on tree grammars



1. Given L, compute G s.t. $L(G) = L$ (or $L(G) \supseteq L$).
2. Solve second-order unification problem, e.g.

$$S(X) \quad \equiv \quad X(\alpha) \to (X(t_1) \wedge X(t_2)) \vdash A(u_1), A(u_2), A(u_3)$$

Find F s.t. $S(F)$ is a tautology

- Quantifier-free
- Every solution gives proof with cuts
- For $S(X)$ induced by grammar for Π_1-cuts: always solvable

Outline

- ✓ Cut-elimination and cut-introduction
- ⇒ Inductive theorem proving based on tree grammars
- ▶ Example

Cut and induction

$$\dfrac{\begin{array}{cc}(\pi_b) & (\pi_s(\alpha)) \\ \Gamma \vdash A(0) & \Gamma, A(\alpha) \vdash A(s\alpha)\end{array}}{\Gamma \vdash A(t)}\ \text{ind}$$

If t is variable-free, there is $n \in \mathbb{N}$ s.t. $|t| = n$

$$\dfrac{\dfrac{\begin{array}{cc}(\pi_1) & (\pi_2(0)) \\ \Gamma \vdash A(0) & A(0), \Gamma \vdash A(s0)\end{array}}{\Gamma \vdash A(s0)}\ \text{cut}}{\vdots}$$

$$\dfrac{\begin{array}{cc}\Gamma \vdash A(s^n 0) & A(s^n 0) \vdash A(t)\end{array}}{\Gamma \vdash A(t)}\ \text{cut}$$

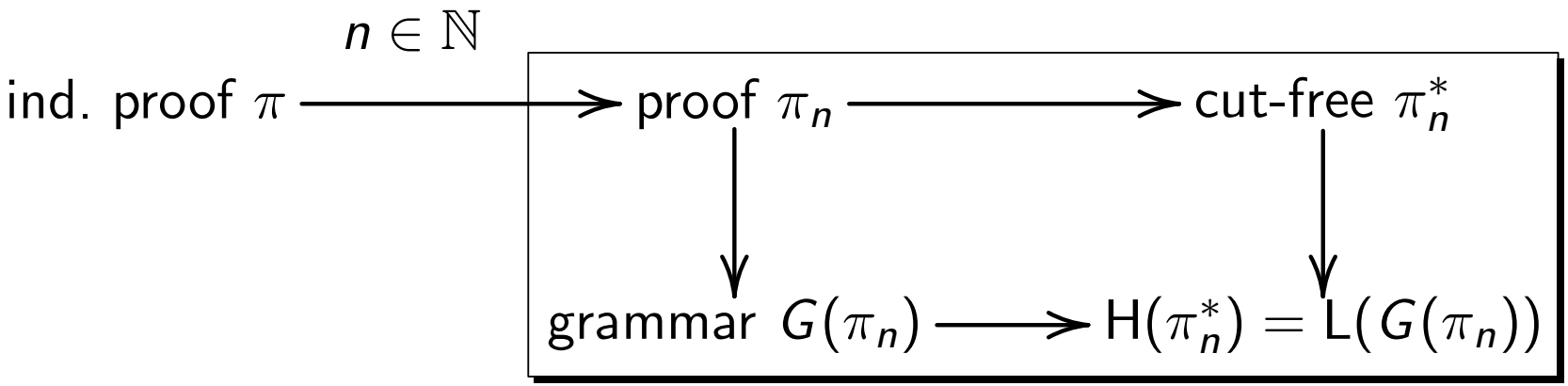
$\Rightarrow$ Induction is infinitary cut

Simple induction proofs

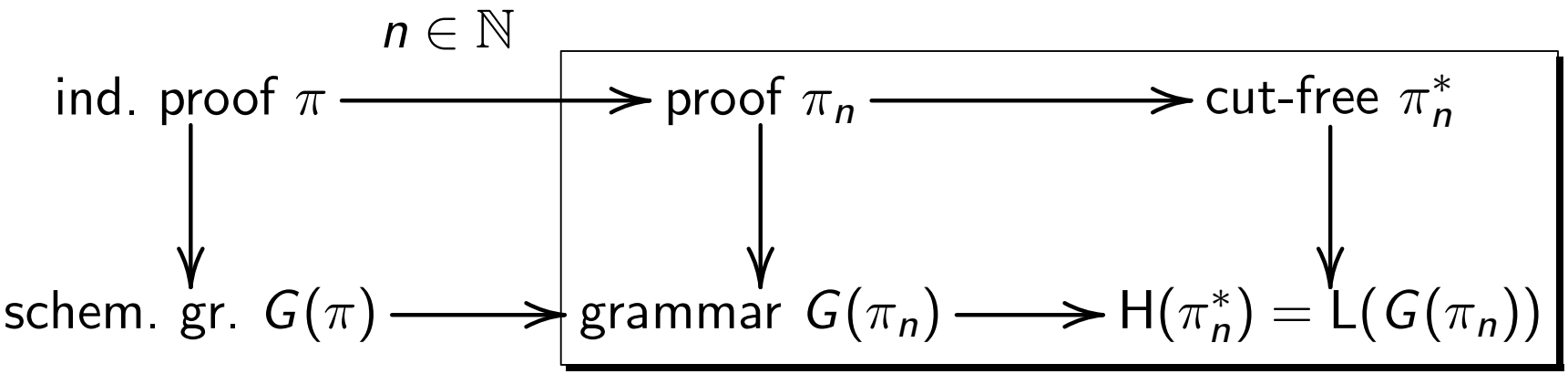
no nested induction, Π_1-invariant

$$\dfrac{\dfrac{\Gamma \vdash \forall y F[\alpha, 0, y] \quad \Gamma, \forall y F[\alpha, \nu, y] \vdash \forall y F[\alpha, s\nu, y]}{\Gamma \vdash \forall y F[\alpha, \alpha, y]} \qquad \Gamma, \forall y F[\alpha, \alpha, y] \vdash B[\alpha]}{\dfrac{\Gamma \vdash B[\alpha]}{\Gamma \vdash \forall x B[x]}}\ \text{cut}$$

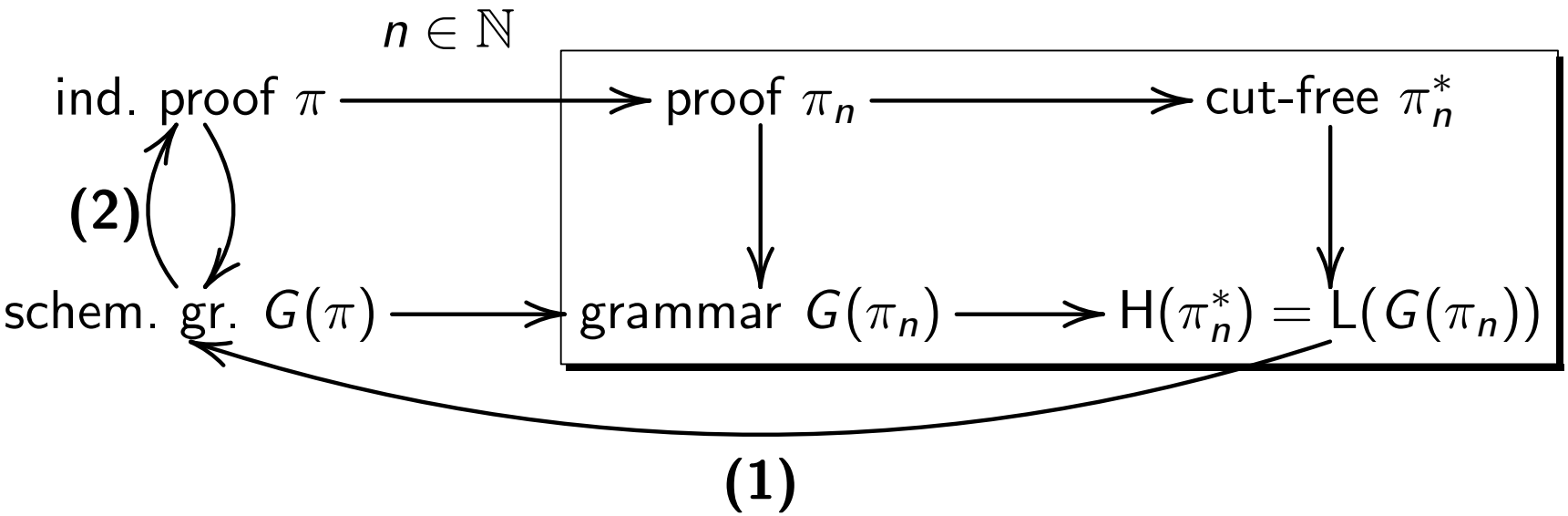
Induction and tree grammars (1/3)



Induction and tree grammars (2/3)



Induction and tree grammars (3/3)



Inductive theorem proving based on tree grammars

1. Compute schematic grammar G from $\{H(\pi_i^*) \mid i \in M \subseteq_{\text{fin}} \mathbb{N}\}$ s.t. $\mathsf{L}(G_i) \supseteq H(\pi_i^*)$ for all $i \in M$.
   - Finding smallest such G by PTIME-translation to MaxSAT
   - No coverage-guarantee for all $i \in \mathbb{N}$

2. Solve unification problem of the form:
   $\Gamma_0 \vdash X[\alpha, 0, \beta]$
   $\Gamma_1, \bigwedge_{1 \leq i \leq n} X[\alpha, \nu, t_i[\alpha, \nu, \gamma]] \vdash X[\alpha, s\nu, \gamma]$
   $\Gamma_2, \bigwedge_{1 \leq i \leq m} X[\alpha, \alpha, u_i[\alpha]] \vdash B[\alpha]$
   - Quantifier-free
   - Every solution gives inductive proof
   - In general: undecidable, but complete algorithm
   - Fast heuristics

The algorithm

Input: background theory Γ, statement $\forall x\, A(x)$

1. Compute instance proofs $\{\pi_i^* : A(i) \mid i \in M \subseteq_{\text{fin}} \mathbb{N}\}$
2. Compute schematic grammar G s.t. $L(G_i) \supseteq H(\pi_i^*)$ for $i \in M$
3. $S(X) :=$ unification problem induced by G
4. (Try to) find an F s.t. $S(F)$ is a tautology
5. $\pi :=$ proof induced by $S(F)$

Output: inductive proof π of $\Gamma \vdash \forall x\, A(x)$

Outline

- ✓ Cut-elimination and cut-introduction
- ✓ Inductive theorem proving based on tree grammars
- ⇒ Example

Example: Factorial (1/4)

- Background theory

$f(0) = 1$	$(f0)$
$f(sx) = sx \cdot f(x)$	$(fST(x))$
$g(x, 0) = x$	$(g0(x))$
$g(x, sy) = g(x \cdot sy, y)$	$(gST(x, y))$
$x \cdot 1 = x$	$(1R(x))$
$1 \cdot x = x$	$(1L(x))$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	$(ASSO(x, y, z))$

- Want to prove $g(1, \alpha) = f(\alpha)$
- Needs generalisation to $\forall y\, g(y, \alpha) = y \cdot f(\alpha)$

Example: Factorial (2/4)

- Instance-proof for $n = 2$:

$$\begin{aligned} g(1,2) &= g(1 \cdot 2, 1) = g((1 \cdot 2) \cdot 1, 0) = (1 \cdot 2) \cdot 1 \\ &= ((1 \cdot 2) \cdot 1) \cdot 1 = ((1 \cdot 2) \cdot 1) \cdot f(0) \\ &= (1 \cdot 2) \cdot (1 \cdot f(0)) = (1 \cdot 2) \cdot f(1) \\ &= 1 \cdot (2 \cdot f(1)) = 1 \cdot f(2) = f(2) \end{aligned}$$

- Uses the axiom-instances:
  $gST(1,1), gST(1 \cdot 2, 0), g0((1 \cdot 2) \cdot 1)$
  $fST(1), fST(0), f0$
  $ASSO(1 \cdot 2, 1, f(0)), ASSO(1, 2, f(1))$
  $\ldots$
- Analogous for other n

Example: Factorial (3/4)

- Schematic grammar

$$\begin{aligned}
\tau &\to \mathsf{r}_{f0} \mid \mathsf{r}_{1L}(f(\alpha)) \mid \mathsf{r}_{g0}(\beta) \mid \mathsf{r}_{1R}(\beta) \\
&\quad\ \mathsf{r}_{fST}(\nu) \mid \mathsf{r}_{gST}(\gamma, \nu) \mid \mathsf{r}_{ASSO}(\gamma, s\nu, f(\nu)) \\
\gamma &\to \gamma \cdot s\nu \\
\gamma_{end} &\to 1
\end{aligned}$$

obtained from instance proofs for $M = \{2, 3\}$.

Example: Factorial (4/4)

- Unification problem:

$$f(0) = 1, 1 \cdot f(\alpha) = f(\alpha), g(\beta, 0) = \beta, \beta \cdot 1 = \beta \vdash X[\alpha, 0, \beta]$$

$$f(0) = 1, 1 \cdot f(\alpha) = f(\alpha), f(s\nu) = s\nu \cdot f(\nu), g(\gamma, s\nu) = g(\gamma \cdot s\nu, \nu),$$
$$(\gamma \cdot s\nu) \cdot f(\nu) = \gamma \cdot (s\nu \cdot f(\nu)), X[\alpha, \nu, \gamma \cdot s\nu] \vdash X[\alpha, s\nu, \gamma]$$

$$f(0) = 1, 1 \cdot f(\alpha) = f(\alpha), X[\alpha, \alpha, 1] \vdash g(1, \alpha) = f(\alpha)$$

- Heuristic
  - Start from C_0 :
    $f(0) = 1 \wedge 1 \cdot f(\alpha) = f(\alpha) \wedge g(\beta, 0) = \beta \wedge \beta \cdot 1 = \beta$
  - Forgetful paramodulation, generalization $0 \mapsto \alpha$,
    forgetful resolution (N/A here)
  - Search space of $\sim$ 20 quasi-tautology checks

  $\Rightarrow$ yields $g(\gamma, \alpha) = \gamma \cdot f(\alpha)$

Summary

Gist of this approach:

- Reduction of problem with quantifiers to one without
  - Compute schematic grammar based on ground terms only
  - Schematic grammar fixes many aspects of induction proof

State of theory:

- Cut-elimination: Π_2-cuts
- Cut-introduction: Π_1-cuts
- Induction: a single Π_1-induction

State of implementation (`http://www.logic.at/gapt/`):

- Introduction of one Π_1-cut: done
  (empirical evaluation in IJCAR '14)
- Introduction of Π_1-cuts: almost done
- Induction: starting now